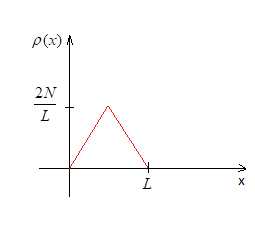
**Special Relativity Dust/Fluid Dynamics**

**An Aside on Lorentz Transformations of Fields**

So far we have been saying that things transform covariantly, which is to mean that they are constructed the same way in each coordinate system – no system is privaleged. So far this has meant that we just take {x,t} → {x´,t´}. But this is to facile a prescription for fields. Consider a 1D frame in which N particles of matter is at rest, and distributed according to the function below.



So we could say:



and if we wanted to calculate the current density, as in j = ρv, then we would say:



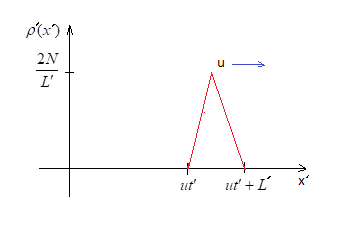
But then say we’re in a frame instantaneously aligned with the origin at t = 0, but moving with speed u to the left. Then what is ρ´(x´,t´)? It isn’t:



even at t = 0. And j´(x´,t´) is certainly not:



So in that sense, we cannot say that the density field transforms ‘covariantly’. But I guess it would transform in the same way as in we would *measure* it the same way – count the number of particles within an interval dx´ and call that ratio ρ´(x´). And measure v´(x´,t´) likewise, to get j´(x´,t´). So then considering length contraction, whereby L´ = L/γ,



we should find, where ρ´max = ρmax/γ.



and,



Compare this to:



so first we have to solve for the relationship between (ct´, x´) and (ct, x). This is:



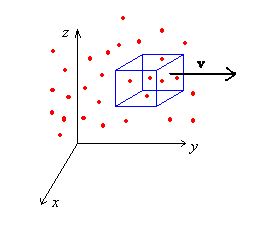
and so,



which checks out. So let’s observe, then, that a classical field such as this does not transform covariantly in the sense of ‘same functional form’. It almost transforms as a scalar, meaning the function values are the same at a given point P, regardless of what coordinate system it’s expressed in. But due to length contraction it does not even do this. General rule is simply that it is covariant in the sense that it will be ‘constructed’ the same way.

**Dust**

So let’s proceed first with the simplest case – a system of non-interacting particles, each moving with the same velocity. This is called dust. We take dust to mean a collection of particles moving around in space at some collection of velocities. We will focus our considerations on some sub-volume of the dust, moving with velocity **v** with respect to our rest frame.



Let the density of these particles be ρ = m/(Δx·Δy·Δz), where m is taken to be the total rest mass of course (and m will always stand for this quantity).

**Continuity Equation**

The change in number of particles in the box is equal to the rates at which these particles exit the box (assuming that there is matter-energy conversions going on). This brings us to the continuity equation, which can be developed exactly as before in the context of fluids.



where n is the density of particles in the frame in which they’re being observed.

*Relativistically Invariant Formulation of Continuity Equation*

We can write this as:



We can put jμ in manifestly invariant form. We just recognize that the density is n = γn0 where n0 is the density in the rest frame of the particles/box, since by length contraction, the volume of the box will shrink by factor γ. So then we can write:



and so our relativistic continuity equation – which says the same thing as the classical continuity equation – is:



where we define the space-time current .

**N2L and WE equations**

Now let’s talk about the dynamics of our dust particles. First we’ll look at N2L And then we’ll talk about the WE equation. These two we’ll be able to combine togther into a manifestly invariant form. Apropos the first, we have that the rate of change of momentum in the fluid is equal to the rate at which it is exiting the fluid through the boundaries, plus the rate at which it is increasing due to forces present. We’ll ignore the possibility of external forces, and so we’ll just have that the rate of change of momentum in some closed volume is equal to the momentum flux:



To obtain a differential relationship, we just convert the surface integrals to volume integrals,



We’re going to make it prettier and perhaps more intuitive. Let’s make the following definitions.



Then we can write N2L in differential form as:



Now let’s look at what the work-energy equation has to say. So the energy content of a fluid is the sum of its internal and potential energies. This energy can change depending on how many and energetic particles are entering or leaving the space are, and also what non-conservative forces, heat exchange is going on between the system and its external environment. For our dust, we’ll ignore the possibility of heat exchange, external forces, or viscous forces, as before. So putting all of these statements into math, we simply have that the rate of energy change within the volume is equal to the energy flux.



So then, converting everything to volume integrals…



we have in differential form,



Now we define the internal energy, and energy flux vector, again:



Then we can write:



*Relativistically Invariant Formulation of N2L + WE equation I*

Desiring to formulate the equations of motion in a relativistically invariant way, we can combine these equations as:



and so we have:



Where Tαβ defines the stress-energy tensor. Explicitly, the stress-energy tensor is:



And we can put this in explicitly relativistically invariant form if we recognize that the density of particles ρ, differs from the density of particles in the rest frame ρ0 by ρ = γρ0. The contraction occurs because we can decompose the box into little tiny boxes, all with their front faces parallel to **v**. Then each box will length contract by an amount γ = 1/√[1-(v/c)2]. And so the whole will contract by the same amount. And so the density will increase by that amount. So filling this in we have:



or writing this in terms of space-time vectors:



*Relativistically Invariant Formulation of N2L + WE equation II*

There is another way to do this which will prove useful in a moment. We proceed backwards. Let’s start with the definition of the stress-energy tensor [μ = 0,i; ν = 0,j]



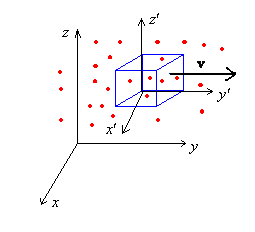
Now that we have the stress-energy tensor defined, it should be relatively clear that by definition:



because written out this says,



which is just the continuity equation for energy and momentum flow. These equations are true almost by definition. So with that acknowledged, we can go a simpler route to finding **T**, and thence the equations of motion. The basic idea is to let relativity work for us, rather than against us, by simply writing out what **T** is in a reference in which the sub-volume of dust is stationary (observe that our equations above just involve the local density and local velocity; so we can go to a local rest frame).



And then using a Lorentz transformation to get it in a frame in which it is moving with speed **v**. So in the rest frame we have:



since the energy density in the rest frame is ρc2γ = ρ0c2(1). Also, since its in the rest frame, it is stationary and so there is no momentum density. Also there is no flux of energy across the surface because we assume no heat conduction, and further more that there are no particles moving through the surface. Finally there is no flux of momentum because there are no forces at the surface, and no particles carrying momentum moving through the surface – in that frame. So this is the tensor. Alright, but now how do we get it for when the dust is moving to the right with speed v? We do it by simply moving to a reference frame S′ which is moving to the left of the dust with velocity –v. Then with respect to that reference frame S′, the dust will be moving to the right with speed v. The stress-energy tensor, being a tensor, will transform according to:



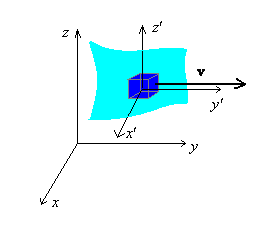
and so this is just a triple matrix product. And we have (well it is kind of ugly looking, though easy to do)



which is the same as we had before.

**Fluids**

Now let’s move on to fluids. Fluids are a little different than dust in that there are inter-particle forces in the form of pressure (honestly I don’t know why dust doesn’t have pressure too – when we examine general fluids in stat mech or thermodynamics, we see a pressure term naturally arises due to fluctuations of the velocity about the mean – maybe there are no fluctuations for dust? Yes, that is the distinction. There are no fluctuations of the velocity about the mean.). So consider a fluid moving around through space. Again we will focus on a sub-volume, high-lighted in blue, moving at (perhaps just momentarily) constant velocity **v**.



**Continuity Equation**

First let’s recognize that we have the usual continuity equation for the fluid as well as for the dust:



**N2L and WE Equations**

Now let’s look at the dynamics of the fluid (assuming non-viscous and incompressible for simplicity). We’ll write the stress-energy of this section of the fluid in its stationary reference frame S′. Then we will do a Lorentz transformation to a reference frame in which it is moving with velocity **v** to get the general stress energy tensor. And then once we have that, we can get the equations of motion. You may wonder why we must do it like this, since we actually worked out N2L and the WE equation for fluids a few lectures ago. But there are some subtleties in the relativistic analysis of fluid motion which aren’t obvious (to me anyway) and taking this seemingly straightforward approach doesn’t yield the correct equations. So better to do it the safer way, where we know relativity isn’t a factor (because we’re starting in the rest frame where our classical intuition is safe). So in the rest frame we have (again μ = 0,i; ν = 0,j):



The explanation for this is as follows. Again the energy density is simply the rest energy (density). There is no momentum density because everything is at rest in this frame. There is no energy flux across the surface of the interior box because there is no heat conduction, and no particles flowing inside or out. There is transfer of momentum though, because there are pressure forces acting on the surfaces. Tij = Pδij here because the force on the ith component of the force on the surface area pointing in the jth direction is 0 unless i = j. In other words, all forces are normal to the surfaces. So this gives us Tu′υ′. Now let’s boost this tensor by Λ(-**v**) to get Tμυ valid in the reference frame, w/r to which the fluid is moving with velocity **v**. We have (ε0 is rest energy density):



which comes to:



and then, after laborious calculation goes to:



and so we can write the tensor itself as:



where is the metric. Might note that setting P = 0, we get the dust tensor. Let’s go on to derive the relativistic equations of motion of our fluid. Since we’ve ignored the possibility of viscosity or external forces, we ought to get something on the order of the Navier-Stokes equations, but with **f** and λ set to 0. Interestingly, though, there will be relativistic corrections. OK, now due to what the terms in **T** represent, we argued that it must be true that:



So let’s work out the components of this equation. First we’ll look at the 0th component.



Using the continuity equation, after a little bit of work the υ = 0 component of this equation yields



and the ν = i components of the equation yield the relativistic Bernoulli equation equation:



If you look back a few lectures, you’ll see that the coefficient in front of the derivatives on v was just ρ, the mass density. And that is what our coefficient here reduces to in the non-relativistic limit where ε0 = mc2 >> P.